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Algorithmes à convergence ultra-rapide !
Basés sur la théorie des équations modulaires ou sur la moyenne arithmético-géométrique
(XXe siècle)

Salamin/Brent :

1976 : $a_0 = 1 \qquad b_0 = \frac{1}{\sqrt{2}} \qquad a_{n+1} = \frac{a_n + b_n}{2} \qquad b_{n+1} = \sqrt{a_n b_n}$

$$U_m = \frac{4a_m^2}{1 - 2\sum_{n=1}^{m} 2^n (a_n^2 - b_n^2)} \xrightarrow{m \to \infty} \pi$$

variante facile à construire :

$$a_0 = 1 \qquad b_0 = \frac{1}{\sqrt{2}} \qquad u_0 = 0 \qquad v_0 = 1$$

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad b_{n+1} = \sqrt{a_n b_n} \qquad u_{n+1} = \frac{u_n + v_n}{2} \qquad v_{n+1} = \frac{u_n b_n + a_n v_n}{2 b_{n+1}}$$

$$2\sqrt{2}\,\frac{a_n^3}{u_n} \overset{n \to \infty}{\sim} 2\sqrt{2}\,\frac{a_n^3}{v_n} \overset{n \to \infty}{\sim} 2\sqrt{2}\,\frac{b_n^3}{u_n} \overset{n \to \infty}{\sim} 2\sqrt{2}\,\frac{b_n^3}{v_n} \xrightarrow{n \to \infty} \pi$$

Je n'ai pas encore testé les algorithmes qui suivent.

Convergence quadratique

$$f(k) = k.2^{-\frac{k}{4}}\left[\sum_{n=1}^{\infty} 2^{-k\frac{n(n-1)}{2}}\right]^2 \qquad g_0 = \frac{f(n)}{f(2n)} \qquad g_k = \sqrt{\frac{1}{2}\left(g_{k-1} + \frac{1}{g_{k-1}}\right)}$$

$$\pi = 2\ln(2) f(n) \prod_{k=1}^{\infty} g_k$$

Convergence cubique

Cet algorithme converge vers le plus proche multiple de *Pi* de f_0.

$f_n=f_{n-1}+\sin(f_{n-1})$

J et P.Borwein :

1) *1984* : convergence quadratique (reposant sur la moyenne arithmético-géométrique)

$$a_0=\sqrt{2}\qquad b_0=0\qquad a_{n+1}=\frac{\sqrt{a_n}}{2}+\frac{1}{2\sqrt{a_n}}\qquad b_{n+1}=\frac{\sqrt{a_n}\,(1+b_n)}{a_n+b_n}$$

$$p_0=2+\sqrt{2}\qquad p_{n+1}=p_n b_{n+1}\frac{1+a_{n+1}}{1+b_{n+1}}\xrightarrow{n\to\infty}\pi$$

2) *1987* : convergence quadratique (reposant aussi sur l'*AGM*)

$$y_0=\sqrt{2}\qquad z_1=\sqrt[4]{2}\qquad y_{n+1}=\frac{1+y_n}{2\sqrt{y_n}}\qquad z_{n+1}=\frac{1+y_n z_n}{(1+z_n)\sqrt{y_n}}$$

$$f_0=2+\sqrt{2}\qquad f_n=f_{n-1}\frac{1+y_n}{1+z_n}\xrightarrow{n\to\infty}\pi$$

3) convergence quadratique (reposant sur les équations modulaires comme les suivantes)

$$y_0=\frac{1}{\sqrt{2}}\qquad y_{n+1}=\frac{1-\sqrt{1-y_n^2}}{1+\sqrt{1-y_n^2}}\qquad \alpha_0=\frac{1}{2}\qquad \alpha_{n+1}=\left((1+y_{n+1})^2\alpha_n\right)-2^{n+1}y_{n+1}$$

$$\beta_n=\frac{1}{\alpha_n}\xrightarrow{n\to\infty}\pi$$

$$y_0=\frac{1}{3}\qquad y_{n+1}=\frac{1-\sqrt{1-y_n^2}}{1+3\sqrt{1-y_n^2}}\qquad \alpha_0=\frac{1}{3}\qquad \alpha_{n+1}=(1+3y_{n+1})\alpha_n-2^n y_{n+1}$$

$$\beta_n=\frac{1}{\alpha_n}\xrightarrow{n\to\infty}\pi$$

4) convergence quadratique :

$$y_0=2\qquad y_n=\frac{4}{1+\sqrt{(4-y_{n-1})(2+y_{n-1})}}\qquad \alpha_0=\frac{1}{3}\qquad \alpha_n=y_{n-1}\alpha_{n-1}+\frac{2^{n-1}}{3}(1-y_{n-1})$$

$$\beta_n=\frac{1}{\alpha_n}\xrightarrow{n\to\infty}\pi$$

5) convergence cubique :

$$y_1 = \frac{\sqrt{3}-1}{2} \quad y_n = \frac{1-\sqrt[3]{1-y_{n-1}^3}}{1+2\sqrt[3]{1-y_{n-1}^3}} \quad \alpha_0 = \frac{1}{3} \quad \alpha_n = \left((1+2y_n)^2\,\alpha_{n-1}\right) - 4.3^{n-2}(1+y_n)y_n$$

$$\beta_n = \frac{1}{\alpha_n} \xrightarrow{n\to\infty} \pi$$

6) convergence quartique :

$$y_1 = \sqrt{2}-1 \quad y_{n+1} = \frac{1-\sqrt[4]{1-y_n^4}}{1+\sqrt[4]{1-y_n^4}} \quad \alpha_0 = 6-4\sqrt{2} \quad \alpha_{n+1} = \left((1+y_{n+1})^4\,\alpha_n\right) - 2^{2n+3}\left(1+y_{n+1}+y_{n+1}^2\right)y_{n+1}$$

$$\beta_n = \frac{1}{\alpha_n} \xrightarrow{n\to\infty} \pi$$

7) convergence quintique :

$$S_0 = 5\left(\sqrt{5}-2\right) \quad \alpha_0 = \frac{1}{2} \quad S_{n+1} = \frac{25}{S_n\left(Z+\frac{X}{Z}+1\right)^2}$$

$$\text{où} \quad X = \frac{5}{S_n}-1 \quad Z = \frac{1}{2}\sqrt[5]{X\left(Y+\sqrt{Y^2-4X^3}\right)} \quad Y = (X-1)^2+7$$

$$\alpha_{n+1} = S_n^2\alpha_n - 5^n\left(\frac{S_n^2-5}{2}+\sqrt{S_n(S_n^2-2S_n+5)}\right) \quad \beta_n = \frac{1}{\alpha_n} \xrightarrow{n\to\infty} \pi$$

8) convergence septique :

$$\alpha_0 = \frac{4}{3\sqrt{7}} \quad M = \left(2\cos\left(\frac{4\pi ij}{7}\right)\right)_{1\le i,j\le 3} \quad x_2 < x_1 < x_3 \text{ solutions de } 27^4x^3 - 27^3 32x^2 + 27^2 325x - 13^4 = 0$$

$$y_1 = \left(x_1^3x_3\right)^{\frac{1}{7}} \quad y_2 = \left(x_2^3x_1\right)^{\frac{1}{7}} \quad y_3 = \left(x_3^3x_2\right)^{\frac{1}{7}} \quad s_0 = \left(\frac{27}{13}\right)^{\frac{3}{7}}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad s_n = \frac{1}{7}\sqrt{m_{n-1}}\left[M s^\times_{n-1} + 1\right]$$

$$g_1 = s_{1,n}s_{2,n}s_{3,n} \quad g_2 = s_{1,n}^3s_{2,n} + s_{2,n}^3s_{3,n} + s_{3,n}^3s_{1,n} \quad g_3 = 1 - \frac{10}{7}g_1 + \frac{1}{7}g_2 \quad g_4 = 3 - \frac{51}{7}g_1 + \frac{10}{7}g_2$$

$$m_n = \frac{49}{\left(1 + 2s^\times_n\begin{pmatrix}1\\1\\1\end{pmatrix}\right)} \quad s^\times_n = \begin{pmatrix} \left(\frac{\mu^3\gamma}{g_3^3}\right)^{\frac{1}{7}} \\ \left(\frac{\beta^3\mu}{g_3^3}\right)^{\frac{1}{7}} \\ \left(\frac{\gamma^3\beta}{g_3^3}\right)^{\frac{1}{7}} \end{pmatrix} \quad \beta < \mu < \gamma \text{ solutions de } x^3 - g_4x^2 + xg_3(2g_4 - 3g_3) - g_3^4 = 0$$

$$\alpha_n = m_{n-1}\alpha_{n-1} + \sqrt{7}\,\frac{7^{n-1}}{3}(1 - m_{n-1}) \xrightarrow{\infty} \frac{1}{\pi}$$

9) convergence nonique :

$$\alpha_0 = \frac{1}{3} \quad s^\times_1 = \frac{\sqrt{3}-1}{2} \quad s_1 = \left(1 - \left(s^\times_1\right)^3\right)^{\frac{1}{3}} \quad s_{n+1} = \frac{(1 - s^\times_n)^3}{(t + 2u)(t^2 + tu + u^2)}$$

$$\text{avec} \quad t = 1 + 2s^\times_n \quad u = \left[9s^\times_n(1 + s^\times_n + (s^\times_n)^2)\right]^{\frac{1}{3}} \quad s^\times_n = \left(1 - \left(s_n\right)^3\right)^{\frac{1}{3}}$$

$$m = 27\,\frac{(1 + s_n + s_n^2)}{t^2 + tu + u^2} \quad \alpha_n = m\alpha_{n-1} + 3.9^{n-2}(1 - m) \xrightarrow{n\to\infty} \frac{1}{\pi}$$

10) convergence "hexadécimalique" ! (ordre *16*)

$$\alpha_0 = \frac{1}{3} \quad s_1 = \sqrt{2} - 1 \quad s^\times_1 = \left(1 - \left(s_1\right)^4\right)^{\frac{1}{4}} \quad s_{n+1} = \frac{(1 - s^\times_n)^4}{(t + u)^2(t^2 + u^2)} \text{ avec}$$

$$m_1 = \left(\frac{1 + s_n}{t}\right)^4 \quad m_2 = \frac{1}{t^4} \quad t = 1 + s^\times_n \quad u = \left[8s^\times_n(1 + (s^\times_n)^2)\right]^{\frac{1}{4}} \quad s^\times_n = \left(1 - s_n^4\right)^{\frac{1}{4}}$$

$$\alpha_n = 16m_1\alpha_{n-1} + \frac{4^{2n-1}}{3}(1 - 12m_2 - 4m_1) \xrightarrow{n\to\infty} \frac{1}{\pi}$$

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